Spanning trees on the Sierpinski gasket

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We obtain the numbers of spanning trees on the Sierpinski gasket $SG_d(n)$ with dimension d equal to two, three and four. The general expression for the number of spanning trees on $SG_d(n)$ with arbitrary d is conjectured. The numbers of spanning trees on the generalized Sierpinski gasket $SG_{d,b}(n)$ with $d = 2$ and $b = 3, 4$ are also obtained.

Keywords: Spanning trees, Sierpinski gasket, exact solutions.

I. INTRODUCTION

The enumeration of the number of spanning trees $N_{ST}(G)$ on a graph G was first considered by Kirchhoff in the analysis of electric circuits [1]. It is a problem of fundamental interest in mathematics [2, 3, 4, 5] and physics [6, 7]. The number of spanning trees is closely related to the partition function of the q-state Potts model in statistical mechanics [8, 9]. Some recent studies on the enumeration of spanning trees and the calculation of their asymptotic growth constants on regular lattices were carried out in Refs. [10, 11, 12, 13]. It is of interest to consider spanning trees on self-similar fractal lattices which have scaling invariance rather than translational invariance. Fractals are geometric structures of noninteger Hausdorff dimension realized by repeated construction of an elementary shape on progressively smaller length scales [14, 15]. A well-known example of fractal is the Sierpinski gasket which has been extensively studied in several contexts [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. We shall derive rigorously the numbers of spanning trees on the Sierpinski gasket with dimension equal to two, three and four. The corresponding asymp-

totic growth constants have simple expressions. We shall also conjecture the general expression of the number of spanning trees on the Sierpinski gasket with arbitrary dimension.

II. PRELIMINARIES

We first recall some relevant definitions for spanning trees and the Sierpinski gasket in this section. A connected graph (without loops) $G = (V, E)$ is defined by its vertex (site) and edge (bond) sets V and E [2, 27]. Let $v(G) = |V|$ be the number of vertices and $e(G) = |E|$ the number of edges in G. A spanning subgraph G' is a subgraph of G with the same vertex set V and an edge set $E' \subseteq E$. As a tree is a connected graph with no circuits, a spanning tree on G is a spanning subgraph of G that is a tree and hence $e(G') = v(G) - 1$. The degree or coordination number k_i of a vertex $v_i \in V$ is the number of edges attached to it. A k-regular graph is a graph with the property that each of its vertices has the same degree k. In general, one can associate an edge weight x_{ij} to each edge connecting adjacent vertices v_i and v_j (see, for example [10]). For simplicity, all edge weights are set to one throughout this paper. A well-known method to enumerate spanning trees is to construct the Laplacian matrix $Q(G)$, which is the $v(G) \times v(G)$ matrix with element $Q(G)_{ij} = k_i \delta_{ij} - A(G)_{ij}$ where $A(G)$ is the adjacency matrix. One of the eigenvalues of $Q(G)$ is always zero. Denote the rest as $\lambda(G)_i$, $1 \le i \le v(G) - 1$, then the number of spanning trees $N_{ST}(G) = (1/v(G)) \prod_{i=1}^{v(G)-1} \lambda(G)_i$ [2].

When the number of spanning trees $N_{ST}(G)$ grows exponentially with $v(G)$ as $v(G) \to \infty$, there exists a constant z_G describing this exponential growth [4, 5]:

$$z_G = \lim_{v(G) \to \infty} \frac{\ln N_{ST}(G)}{v(G)} \tag{2.1}$$

where G, when used as a subscript in this manner, implicitly refers to the thermodynamic limit.

For a k-regular graph G_k with $k \ge 3$, there is a upper bound for the number of spanning trees [28, 29]

$$N_{ST}(G_k) \le \left(\frac{2 \ln v(G_k)}{v(G_k) k \ln k} \right) (b_k)^{v(G_k)} , \tag{2.2}$$

where

$$b_k = \frac{(k-1)^{k-1}}{[k(k-2)]^{\frac{k}{2}-1}} . \tag{2.3}$$

With eq. (2.1), this then yields [11]

$$z_{G_k} \leq \ln(b_k) \ . \tag{2.4}$$

The construction of the two-dimensional Sierpinski gasket $SG_2(n)$ at stage n is shown in Fig. 1. At stage $n = 0$, it is an equilateral triangle; while stage $n + 1$ is obtained by the juxtaposition of three n-stage structures. In general, the Sierpinski gaskets SG_d can be built in any Euclidean dimension d with fractal dimensionality $D = \ln(d + 1)/\ln 2$ [17]. For the Sierpinski gasket $SG_d(n)$, the numbers of edges and vertices are given by

$$e(SG_d(n)) = \binom{d + 1}{2}(d + 1)^n = \frac{d}{2}(d + 1)^{n+1} \ , \tag{2.5}$$

$$v(SG_d(n)) = \frac{d + 1}{2}[(d + 1)^n + 1] \ . \tag{2.6}$$

Except the $(d + 1)$ outmost vertices which have degree d, all other vertices of $SG_d(n)$ have degree $2d$. In the large n limit, SG_d is $2d$-regular.
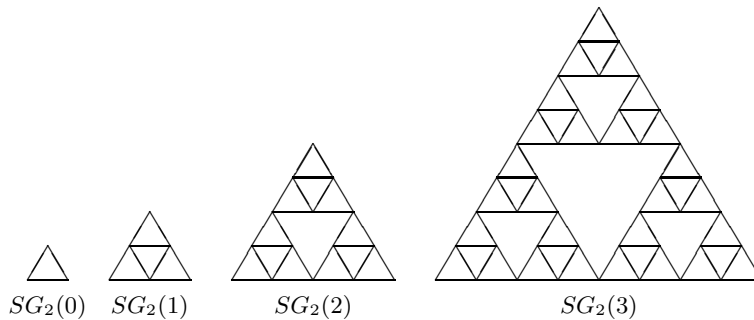

FIG. 1: The first four stages $n = 0, 1, 2, 3$ of the two-dimensional Sierpinski gasket $SG_2(n)$.

The Sierpinski gasket can be generalized, denoted as $SG_{d,b}(n)$, by introducing the side length b which is an integer larger or equal to two [30]. The generalized Sierpinski gasket at stage $n + 1$ is constructed with b layers of stage n hypertetrahedrons. The two-dimensional $SG_{2,b}(n)$ with $b = 3$ at stage $n = 1, 2$ and $b = 4$ at stage $n = 1$ are illustrated in Fig. 2. The ordinary Sierpinski gasket $SG_d(n)$ corresponds to the $b = 2$ case, where the index b is neglected for simplicity. The Hausdorff dimension for $SG_{d,b}$ is given by $D = \ln \binom{b+d-1}{d}/\ln b$ [30]. Notice that $SG_{d,b}$ is not k-regular even in the thermodynamic limit.
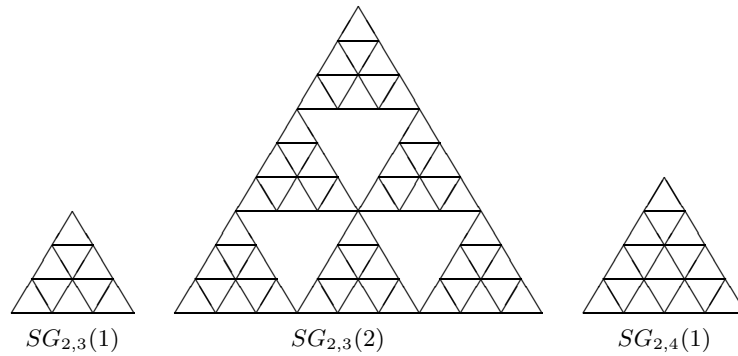
FIG. 2: The generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with $b = 3$ at stage $n = 1, 2$ and $b = 4$ at stage $n = 1$.

III.   THE NUMBER OF SPANNING TREES ON $SG_2(n)$

In this section we derive the number of spanning trees on the two-dimensional Sierpinski gasket $SG_2(n)$ in detail. Let us start with the definitions of the quantities to be used.

Definition III.1 Consider the two-dimensional Sierpinski gasket $SG_2(n)$ at stage n. (a) Define $f_2(n) \equiv N_{ST}(SG_2(n))$ as the number of spanning trees. (b) Define $ga_2(n)$, $gb_2(n)$, $gc_2(n)$ as the number of spanning subgraphs with two trees such that one of the outmost vertices belongs to one tree and the other two outmost vertices belong to the other tree as illustrated in Fig. 3. (c) Define $h_2(n)$ as the number of spanning subgraphs with three trees such that each of the outmost vertices belongs to a different tree.

The quantities $f_2(n)$, $ga_2(n)$, $gb_2(n)$, $gc_2(n)$ and $h_2(n)$ are illustrated in Fig. 3, where only the outmost vertices are shown. It is clear that the values $ga_2(n)$, $gb_2(n)$, $gc_2(n)$ are the same because of rotation symmetry, and we define $g_2(n) \equiv ga_2(n) = gb_2(n) = gc_2(n)$. The initial values at stage 0 are $f_2(0) = 3$, $g_2(0) = 1$, $h_2(0) = 1$. Notice that $h_2(n)$ is the number of spanning trees on $SG_2(n)$ with the three outmost vertices identified. The purpose is to obtain the expression of $f_2(n)$ as follows.

Theorem III.1 The number of spanning trees on the two-dimensional Sierpinski gasket $SG_2(n)$ at stage n is given by

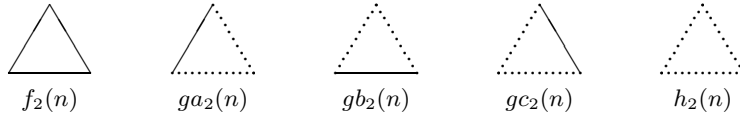$$f_2(n) = 2^{\alpha_2(n)} 3^{\beta_2(n)} 5^{\gamma_2(n)} , \qquad (3.1)$$

FIG. 3: Illustration for the spanning subgraphs $f_2(n)$, $ga_2(n)$, $gb_2(n)$, $gc_2(n)$ and $h_2(n)$. The two outmost vertices at the ends of a solid line belong to one tree, while the two outmost vertices at the ends of a dot line belong to separated trees.

where the exponents are

$$\alpha_2(n) = \frac{1}{2}(3^n - 1) \ , \tag{3.2}$$

$$\beta_2(n) = \frac{1}{4}(3^{n+1} + 2n + 1) \ , \tag{3.3}$$

$$\gamma_2(n) = \frac{1}{4}(3^n - 2n - 1) \ . \tag{3.4}$$

This theorem can be proved by the following two lemmas. The three quantities $f_2(n)$, $g_2(n)$ and $h_2(n)$ satisfy recursion relations which were first obtained in [24].

Lemma III.2 For any non-negative integer n,

$$f_2(n+1) = 6f_2^2(n)g_2(n) \ , \tag{3.5}$$

$$g_2(n+1) = f_2^2(n)h_2(n) + 7f_2(n)g_2^2(n) \ , \tag{3.6}$$

$$h_2(n+1) = 12f_2(n)g_2(n)h_2(n) + 14g_2^3(n) \ . \tag{3.7}$$

Proof The Sierpinski gaskets $SG_2(n+1)$ is composed of three $SG_2(n)$ with three pairs of vertices identified. To obtain the number of spanning trees $f_2(n+1)$, one of the $SG_2(n)$ should be spanned by two trees. There are six possibilities as illustrated in Fig. 4. Therefore, we have

$$f_2(n+1) = 2f_2^2(n)[ga_2(n) + gb_2(n) + gc_2(n)] = 6f_2^2(n)g_2(n) \ . \tag{3.8}$$

Similarly, $ga_2(n+1)$ for $SG_2(n+1)$ can be obtained with appropriated configurations of its three constituting $SG_2(n)$ as illustrated in Fig. 5. Thus,

$$ga_2(n+1) \ = \ f_2^2(n)h_2(n) + 3f_2(n)ga_2^2(n) + 2f_2(n)ga_2(n)gc_2(n)$$

FIG. 4: Illustration for the expression of $f_2(n+1)$.
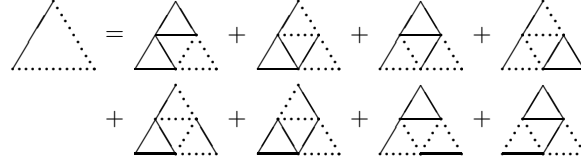
FIG. 5: Illustration for the expression of $ga_2(n+1)$.
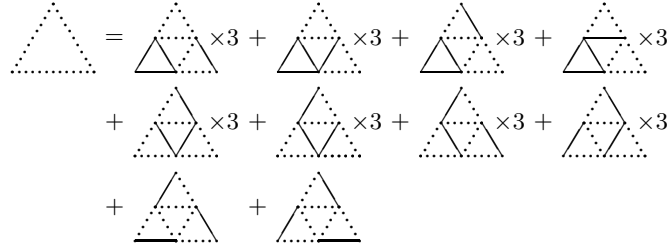
$$+ \ 2f_2(n)ga_2(n)gb_2(n) \ . \tag{3.9}$$

With the identity $ga_2(n) = gb_2(n) = gc_2(n) = g_2(n)$, eq. (3.6) is verified.

Finally, $h_2(n+1)$ is the summation of appropriated configurations as illustrated in Fig. 6, so that

$$h_2(n+1)$$

$$= \ 4f_2(n)h_2(n)[ga_2(n) + gb_2(n) + gc_2(n)]$$

$$+2gc_2(n)ga_2(n)[gc_2(n) + ga_2(n)] + 2ga_2(n)gb_2(n)[ga_2(n) + gb_2(n)]$$

$$+2gb_2(n)gc_2(n)[gb_2(n) + gc_2(n)] + 2ga_2(n)gb_2(n)gc_2(n) \ .$$
$$\tag{3.10}$$

With the identity $ga_2(n) = gb_2(n) = gc_2(n) = g_2(n)$, eq. (3.7) is verified.
$\square$

FIG. 6: Illustration for the expression of $h_2(n+1)$. The multiplication for the eight configurations on the right-hand-side corresponds to three possible orientations.

By the three equations (3.5), (3.6) and (3.7), $f_2(n)$, $g_2(n)$ and $h_2(n)$ can be solved.

Lemma III.3 For any non-negative integer n,

$$f_2(n) = 2^{\alpha_2(n)} 3^{\beta_2(n)} 5^{\gamma_2(n)} \ , \tag{3.11}$$

$$g_2(n) = 2^{\alpha_2(n)} 3^{\beta_2(n)-n-1} 5^{\gamma_2(n)+n} \ , \tag{3.12}$$

$$h_2(n) = 2^{\alpha_2(n)} 3^{\beta_2(n)-2n-1} 5^{\gamma_2(n)+2n} \ . \tag{3.13}$$

where $\alpha_2(n)$, $\beta_2(n)$ and $\gamma_2(n)$ are given in eqs. (3.2)-(3.4)

Proof As $\alpha_2(0) = 0$, $\beta_2(0) = 1$ and $\gamma_2(0) = 0$, eqs. (3.11)-(3.13) are correct for $n = 0$. By the recursion relation (3.5), it is easy to obtain

$$f(n + 1) = 2^{3\alpha_2(n)+1} 3^{3\beta_2(n)-n} 5^{3\gamma_2(n)+n} \tag{3.14}$$

where

$$3\alpha_2(n) + 1 = \frac{1}{2}(3^{n+1} - 1) = \alpha_2(n + 1) \ , \tag{3.15}$$

$$3\beta_2(n) - n = \frac{1}{4}(3^{n+2} + 2n + 3) = \beta_2(n + 1) \ , \tag{3.16}$$

$$3\gamma_2(n) + n = \frac{1}{4}(3^{n+1} - 2n - 3) = \gamma_2(n + 1) \ . \tag{3.17}$$

The proof for eq. (3.11) is completed by induction. Similarly, we have

$$g(n + 1) = 2^{3\alpha_2(n)+1} 3^{3\beta_2(n)-2n-2} 5^{3\gamma_2(n)+2n+1} \ , \tag{3.18}$$

$$h(n + 1) = 2^{3\alpha_2(n)+1} 3^{3\beta_2(n)-3n-3} 5^{3\gamma_2(n)+3n+2} \ , \tag{3.19}$$

which verify eqs. (3.12) and (3.13). $\square$

Although the Laplacian matrix $Q(G)$ for $G = SG_2(n)$ can be constructed, it does not look simple to diagonalize. For example, with appropriate order of the vertices,

$$Q(SG_2(1)) = \begin{pmatrix} 4 & -1 & -1 & 0 & -1 & -1 \\ -1 & 4 & -1 & -1 & 0 & -1 \\ -1 & -1 & 4 & -1 & -1 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 \\ -1 & 0 & -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 \end{pmatrix} \tag{3.20}$$

and

$$Q(SG_2(2))$$

$$= \begin{pmatrix}
4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
-1 & -1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 4 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & -1 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 4 & 0 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 4 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}$$

$$(3.21)$$

Denote the non-zero eigenvalues of $Q(SG_2(n))$ as $\lambda(SG_2(n))_i$ for $1 \leq i \leq v(SG_2(n)) - 1$.

Corollary III.1 The product of non-zero eigenvalues of $Q(SG_2(n))$ is given by

$$\prod_{i=1}^{v(SG_2(n))-1} \lambda(SG_2(n))_i = \frac{3}{2}(3^n + 1)f_2(n) ,$$

$$(3.22)$$

where $f_2(n)$ is given in Theorem III.1 and the number of vertices $v(SG_2(n))$ for $SG_2(n)$ is given by eq. (2.6) with $d = 2$.

By the definition in eq. (2.1), we have the following corollary.

Corollary III.2 The asymptotic growth constant for SG_2 is given by

$$z_{SG_2} = \frac{1}{3}\ln 2 + \frac{1}{2}\ln 3 + \frac{1}{6}\ln 5$$

$$\simeq 1.048594856...$$

$$(3.23)$$

This is equivalent to eq. (20) of [25]. In passing, we notice that the number of spanning trees is the same for a planar graph and its dual, so that Theorem III.1 also applies to the dual of $SG_2(n)$, denoted as $SG_2^*(n)$. As SG_2 is 4-regular in the large n limit, we have $z_{SG_2^*} = z_{SG_2}$ [13].

IV. THE NUMBER OF SPANNING TREES ON $SG_{2,b}(n)$ WITH $b = 3, 4$

The method given in the previous section can be applied to the number of spanning trees on $SG_{d,b}(n)$ with larger values of d and b. The number of configurations to be considered increases as d and b increase, and the recursion relations must be derived individually for each d and b. In this section, we consider the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with the number of layers b equal to three and four. For $SG_{2,3}(n)$, the numbers of edges and vertices are given by

$$e(SG_{2,3}(n)) = 3 \times 6^n \ , \tag{4.1}$$

$$v(SG_{2,3}(n)) = \frac{7 \times 6^n + 8}{5} \ , \tag{4.2}$$

where the three outmost vertices have degree two. There are $(6^n - 1)/5$ vertices of $SG_{2,3}(n)$ with degree six and $6(6^n - 1)/5$ vertices with degree four. Define $f_{2,3}(n)$, $g_{2,3}(n)$, $h_{2,3}(n)$ as in Definition III.1 such that the initial values are $f_{2,3}(0) = 3$, $g_{2,3}(0) = 1$, $h_{2,3}(0) = 1$. By the method illustrated in the above section, we obtain following recursion relations for any non-negative integer n.

$$f_{2,3}(n+1) = 18 f_{2,3}^4(n) g_{2,3}(n) h_{2,3}(n) + 142 f_{2,3}^3(n) g_{2,3}^3(n) \ , \tag{4.3}$$

$$\begin{aligned} g_{2,3}(n+1) &= 2 f_{2,3}^4(n) h_{2,3}^2(n) + 77 f_{2,3}^3(n) g_{2,3}^2(n) h_{2,3}(n) \\ &+ 171 f_{2,3}^2(n) g_{2,3}^4(n) \ , \end{aligned} \tag{4.4}$$

$$\begin{aligned} h_{2,3}(n+1) &= 60 f_{2,3}^3(n) g_{2,3}(n) h_{2,3}^2(n) + 564 f_{2,3}^2(n) g_{2,3}^3(n) h_{2,3}(n) \\ &+ 468 f_{2,3}(n) g_{2,3}^5(n) \ . \end{aligned} \tag{4.5}$$

The figures for these configurations are too many to be shown here. By induction as in Lemma III.3, these equations can be solved.

Theorem IV.1 The number of spanning trees on the two-dimensional Sierpinski gasket $SG_{2,3}(n)$ at stage n is given by

$$f_{2,3}(n) = 2^{\alpha_{2,3}(n)} 3^{\beta_{2,3}(n)} 5^{\gamma_{2,3}(n)} 7^{\delta_{2,3}(n)} \ , \tag{4.6}$$

where the exponents are

$$\alpha_{2,3}(n) = \frac{2}{5}(6^n - 1) \ , \tag{4.7}$$

$$\beta_{2,3}(n) = \frac{1}{25}(13 \times 6^n - 15n + 12) , \qquad (4.8)$$

$$\gamma_{2,3}(n) = \frac{1}{25}(3 \times 6^n - 15n - 3) , \qquad (4.9)$$

$$\delta_{2,3}(n) = \frac{1}{25}(7 \times 6^n + 15n - 7) . \qquad (4.10)$$

$g_{2,3}(n)$ and $h_{2,3}(n)$ can also be expressed by these exponents:

$$g_{2,3}(n) = 2^{\alpha_{2,3}(n)}3^{\beta_{2,3}(n)+n-1}5^{\gamma_{2,3}(n)+n}7^{\delta_{2,3}(n)-n} ,$$

$$h_{2,3}(n) = 2^{\alpha_{2,3}(n)}3^{\beta_{2,3}(n)+2n-1}5^{\gamma_{2,3}(n)+2n}7^{\delta_{2,3}(n)-2n} . \qquad (4.11)$$

By the definition in eq. (2.1), we have the following corollary.

Corollary IV.1 The asymptotic growth constant for $SG_{2,3}$ is given by

$$z_{SG_{2,3}} = \frac{2}{7}\ln 2 + \frac{13}{35}\ln 3 + \frac{3}{35}\ln 5 + \frac{1}{5}\ln 7$$

$$\simeq 1.133231895... \qquad (4.12)$$

For $SG_{2,4}(n)$, the numbers of edges and vertices are given by

$$e(SG_{2,4}(n)) = 3 \times 10^n , \qquad (4.13)$$

$$v(SG_{2,4}(n)) = \frac{4 \times 10^n + 5}{3} , \qquad (4.14)$$

where the three outmost vertices have degree two. There are $(10^n - 1)/3$ vertices of $SG_{2,4}(n)$ with degree six, and $(10^n - 1)$ vertices with degree four. Define $f_{2,4}(n)$, $g_{2,4}(n)$, $h_{2,4}(n)$ as in Definition III.1 such that the initial values are $f_{2,4}(0) = 3$, $g_{2,4}(0) = 1$, $h_{2,4}(0) = 1$. We write a computer program to obtain following recursion relations for any non-negative integer n.

$$f_{2,4}(n+1) = 2f_{2,4}^7(n)h_{2,4}^3(n) + 516f_{2,4}^6(n)g_{2,4}^2(n)h_{2,4}^2(n)$$

$$+ \ 5856f_{2,4}^5(n)g_{2,4}^4(n)h_{2,4}(n) + 11354f_{2,4}^4(n)g_{2,4}^6(n) \ (4.15)$$

$$g_{2,4}(n+1) = 82f_{2,4}^6(n)g_{2,4}(n)h_{2,4}^3(n) + 2786f_{2,4}^5(n)g_{2,4}^3(n)h_{2,4}^2(n)$$

$$+ \ 14480 f_{2,4}^4(n) g_{2,4}^5(n) h_{2,4}(n) + 13732 f_{2,4}^3(n) g_{2,4}^7(n) \quad (4.16)$$

$$
\begin{aligned}
h_{2,4}(n+1) \ = \ & 20 f_{2,4}^6(n) h_{2,4}^4(n) + 2388 f_{2,4}^5(n) g_{2,4}^2(n) h_{2,4}^3(n) \\
& + \ 30948 f_{2,4}^4(n) g_{2,4}^4(n) h_{2,4}^2(n) + 83234 f_{2,4}^3(n) g_{2,4}^6(n) h_{2,4}(n) \\
& + \ 42210 f_{2,4}^2(n) g_{2,4}^8(n) \ .
\end{aligned}
\qquad (4.17)
$$

By induction as in Lemma III.3, these equations can be solved.

Theorem IV.2 The number of spanning trees on the two-dimensional Sierpinski gasket $SG_{2,4}(n)$ at stage n is given by

$$f_{2,4}(n) = 2^{\alpha_{2,4}(n)} 3^{\beta_{2,4}(n)} 5^{\gamma_{2,4}(n)} 41^{\delta_{2,4}(n)} 103^{\epsilon_{2,4}(n)} \ , \qquad (4.18)$$

where the exponents are

$$\alpha_{2,4}(n) = \frac{2}{9}(10^n - 1) \ , \qquad (4.19)$$

$$\beta_{2,4}(n) = \frac{1}{3}(10^n + 2) \ , \qquad (4.20)$$

$$\gamma_{2,4}(n) = \frac{1}{9}(10^n - 1) \ , \qquad (4.21)$$

$$\delta_{2,4}(n) = \frac{2}{27}(2 \times 10^n + 9n - 2) \ , \qquad (4.22)$$

$$\epsilon_{2,4}(n) = \frac{2}{27}(10^n - 9n - 1) \ . \qquad (4.23)$$

$g_{2,4}(n)$ and $h_{2,4}(n)$ can also be expressed by these exponents:

$$
\begin{aligned}
g_{2,4}(n) \ = \ & 2^{\alpha_{2,4}(n)} 3^{\beta_{2,4}(n)-1} 5^{\gamma_{2,4}(n)} 41^{\delta_{2,4}(n)-n} 103^{\epsilon_{2,4}(n)+n} \ , \\
h_{2,4}(n) \ = \ & 2^{\alpha_{2,4}(n)} 3^{\beta_{2,4}(n)-1} 5^{\gamma_{2,4}(n)} 41^{\delta_{2,4}(n)-2n} 103^{\epsilon_{2,4}(n)+2n} \ . \quad (4.24)
\end{aligned}
$$

By the definition in eq. (2.1), we have the following corollary.

Corollary IV.2 The asymptotic growth constant for $SG_{2,4}$ is given by

$$
\begin{aligned}
z_{SG_{2,4}} \ = \ & \frac{1}{6} \ln 2 + \frac{1}{4} \ln 3 + \frac{1}{12} \ln 5 + \frac{1}{9} \ln(41) + \frac{1}{18} \ln(103) \\
\simeq \ & 1.194401490...
\end{aligned}
\qquad (4.25)
$$

V.   THE NUMBER OF SPANNING TREES ON $SG_d(n)$ WITH
$d = 3, 4$

In this section, we derive the number of spanning trees on $SG_d(n)$ with $d = 3, 4$. For the three-dimensional Sierpinski gasket $SG_3(n)$, we use the following definitions.

Definition V.1 Consider the three-dimensional Sierpinski gasket $SG_3(n)$ at stage n. (a) Define $f_3(n) \equiv N_{ST}(SG_3(n))$ as the number of spanning trees. (b) Define $g_3(n)$ as the number of spanning subgraphs with two trees such that one of the outmost vertices belongs to one tree and the other three outmost vertices belong to the other tree. (c) Define $h_3(n)$ as the number of spanning subgraphs with two trees such that two of the outmost vertices belong to one tree and the other two outmost vertices belong to the other tree. (d) Define $p_3(n)$ as the number of spanning subgraphs with three trees such that two of the outmost vertices belong to one tree and the other two outmost vertices separately belong to the other trees. (e) Define $q_3(n)$ as the number of spanning subgraphs with four trees such that each of the outmost vertices belongs to a different tree.

The quantities $f_3(n)$, $g_3(n)$, $h_3(n)$, $p_3(n)$ and $q_3(n)$ are illustrated in Fig. 7, where only the outmost vertices are shown. There are four equivalent $g_3(n)$, three equivalent $h_3(n)$, and six equivalent $p_3(n)$. The initial values at stage 0 are $f_3(0) = 16$, $g_3(0) = 3$, $h_3(0) = 1$, $p_3(0) = 1$, $q_3(0) = 1$. Notice that $q_3(n)$ is the number of spanning trees on $SG_3(n)$ with the four outmost vertices identified.

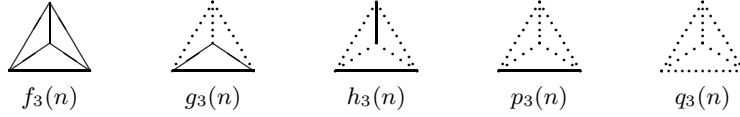$$f_3(n) \qquad g_3(n) \qquad h_3(n) \qquad p_3(n) \qquad q_3(n)$$

FIG. 7: Illustration for the spanning subgraphs $f_3(n)$, $g_3(n)$, $h_3(n)$, $p_3(n)$ and $q_3(n)$. The two outmost vertices at the ends of a solid line belong to one tree, while the two outmost vertices at the ends of a dot line belong to separated trees.

In the process of calculation, we find that it is convenient to combine $g_3(n)$ and $h_3(n)$ and define $gh_3(n) \equiv g_3(n) + h_3(n)$. We obtain following recursion relations for any non-negative integer n.

$$f_3(n + 1) = 72 f_3^2(n) gh_3(n) p_3(n) + 56 f_3(n) gh_3^3(n) , \qquad (5.1)$$

$$gh_3(n + 1) \; = \; 6 f_3^2(n) gh_3(n) q_3(n) + 26 f_3^2(n) p_3^2(n)$$

$$+ \quad 120 f_3(n) gh_3^2(n) p_3(n) + 22 gh_3^4(n) , \tag{5.2}$$

$$p_3(n+1) = 6 f_3^2(n) p_3(n) q_3(n) + 14 f_3(n) gh_3^2(n) q_3(n)$$

$$+ \quad 120 f_3(n) gh_3(n) p_3^2(n) + 88 gh_3^3(n) p_3(n) , \tag{5.3}$$

$$q_3(n+1) = 144 f_3(n) gh_3(n) p_3(n) q_3(n) + 208 f_3(n) p_3^3(n)$$

$$+ \quad 56 gh_3^3(n) q_3(n) + 720 gh_3^2(n) p_3^2(n) . \tag{5.4}$$

The figures for these configurations are too many to be shown here. With the initial values given above, these equations can be solved by induction as in Lemma III.3.

Theorem V.1 The number of spanning trees on the three-dimensional Sierpinski gasket $SG_3(n)$ at stage n is given by

$$f_3(n) = 2^{\alpha_3(n)} 3^{\beta_3(n)} , \tag{5.5}$$

where the exponents are

$$\alpha_3(n) = 4^{n+1} + n , \tag{5.6}$$

$$\beta_3(n) = \frac{1}{3}(4^n - 3n - 1) . \tag{5.7}$$

$gh_3(n)$, $p_3(n)$ and $q_3(n)$ can also be expressed by these exponents:

$$gh_3(n) = 2^{\alpha_3(n)-n-2} 3^{\beta_3(n)+n} ,$$

$$p_3(n) = 2^{\alpha_3(n)-2n-4} 3^{\beta_3(n)+2n} ,$$

$$q_3(n) = 2^{\alpha_3(n)-3n-4} 3^{\beta_3(n)+3n} . \tag{5.8}$$

By the definition in eq. (2.1), we have the following corollary.

Corollary V.1 The asymptotic growth constant for SG_3 is given by

$$z_{SG_3} = 2 \ln 2 + \frac{1}{6} \ln 3$$

$$\simeq 1.569396409... \tag{5.9}$$

For the four-dimensional Sierpinski gasket $SG_4(n)$, we use the following definitions.

Definition V.2 Consider the four-dimensional Sierpinski gasket $SG_4(n)$ at stage n. (a) Define $f_4(n) \equiv N_{ST}(SG_4(n))$ as the number of spanning trees. (b) Define $g_4(n)$ as the number of spanning subgraphs with two trees such that two of the outmost vertices belong to one tree and the other three outmost vertices belong to the other tree. (c) Define $h_4(n)$ as the number of spanning subgraphs with two trees such that one of the outmost vertices belong to one tree and the other four outmost vertices belong to the other tree. (d) Define $p_4(n)$ as the number of spanning subgraphs with three trees such that one of the outmost vertices belong to one tree, two of the other outmost vertices belong to another tree and the rest two outmost vertices belong to the other tree. (e) Define $q_4(n)$ as the number of spanning subgraphs with three trees such that three of the outmost vertices belong to one tree and the other two outmost vertices separately belong to the other trees. (f) Define $r_4(n)$ as the number of spanning subgraphs with four trees such that two of the outmost vertices belong to one tree and the other three outmost vertices separately belong to the other trees. (g) Define $s_4(n)$ as the number of spanning subgraphs with five trees such that each of the outmost vertices belongs to a different tree.

The quantities $f_4(n)$, $g_4(n)$, $h_4(n)$, $p_4(n)$, $q_4(n)$, $r_4(n)$ and $s_4(n)$ are illustrated in Fig. 8, where only the outmost vertices are shown. There are ten equivalent $g_4(n)$, five equivalent $h_4(n)$, fifteen equivalent $p_3(n)$, ten equivalent $q_4(n)$ and ten equivalent $r_4(n)$. The initial values at stage 0 are $f_4(0) = 125$, $g_4(0) = 3$, $h_4(0) = 16$, $p_4(0) = 1$, $q_4(0) = 3$, $r_4(0) = 1$, $s_4(0) = 1$. Notice that $s_4(n)$ is the number of spanning trees on $SG_4(n)$ with the five outmost vertices identified.

We find that it is convenient to reduce the number of variables by defining $gh_4(n) \equiv 3g_4(n) + h_4(n)$ and $pq_4(n) \equiv 2p_4(n) + q_4(n)$. We write a computer program to obtain following recursion relations for any non-negative integer n.

$$
\begin{aligned}
f_4(n+1) \; = \; & 1440 f_4^2(n) gh_4(n) pq_4(n) r_4(n) + 520 f_4^2(n) pq_4^3(n) \\
& + \; 1120 f_4(n) gh_4^3(n) r(n) + 3600 f_4(n) gh_4^2(n) pq_4^2(n) \\
& + \; 1320 gh_4^4(n) pq_4(n) \;, \qquad\qquad\qquad\qquad (5.10)
\end{aligned}
$$

$$
gh_4(n+1) \; = \; 72 f_4^2(n) gh_4(n) pq_4(n) s_4(n) + 378 f_4^2(n) gh_4(n) r_4^2(n)
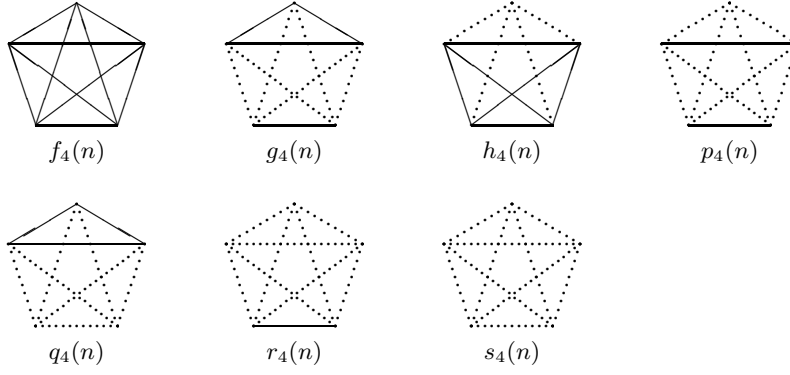$$

FIG. 8: Illustration for the spanning subgraphs $f_4(n)$, $g_4(n)$, $h_4(n)$, $p_4(n)$, $q_4(n)$, $r_4(n)$ and $s_4(n)$. The two outmost vertices at the ends of a solid line belong to one tree, while the two outmost vertices at the ends of a dot line belong to separated trees.

$$+ \ 816f_4^2(n)pq_4^2(n)r_4(n) + 56f_4(n)gh_4^3(n)s_4(n)$$

$$+ \ 3756f_4(n)gh_4^2(n)pq_4(n)r_4(n) + 2360f_4(n)gh_4(n)pq_4^3(n)$$

$$+ \ 688gh_4^4(n)r_4(n) + 2562gh_4^3(n)pq_4^2(n) \ , \tag{5.11}$$

$$pq_4(n+1) \ = \ 48f_4^2(n)gh_4(n)r_4(n)s_4(n) + 52f_4^2(n)pq_4^2(n)s_4(n)$$

$$+ \ 544f_4^2(n)pq_4(n)r_4^2(n) + 240f_4(n)gh_4^2(n)pq_4(n)s_4(n)$$

$$+ \ 1252f_4(n)gh_4^2(n)r_4^2(n) + 4720f_4(n)gh_4(n)pq_4^2(n)r_4(n)$$

$$+ \ 724f_4(n)pq_4^4(n) + 44gh_4^4(n)s_4(n)$$

$$+ \ 3416gh_4^3(n)pq_4(n)r_4(n) + 3104gh_4^2(n)pq_4^3(n) \ , \tag{5.12}$$

$$r_4(n+1) \ = \ 72f_4^2(n)pq_4(n)r_4(n)s_4(n) + 126f_4^2(n)r_4^3(n)$$

$$+ \ 168f_4(n)gh_4^2(n)r_4(n)s_4(n) + 360f_4(n)gh_4(n)pq_4^2(n)s_4(n)$$

$$+ \ 3756f_4(n)gh_4(n)pq_4(n)r_4^2(n) + 2360f_4(n)pq_4^3(n)r_4(n)$$

$$+ \ 264gh_4^3(n)pq_4(n)s_4(n) + 1376gh_4^3(n)r_4^2(n)$$

$$+ \ 7686gh_4^2(n)pq_4^2(n)r_4(n) + 2328gh_4(n)pq_4^4(n) \ , \tag{5.13}$$

$$s_4(n+1) \ = \ 2880f_4(n)gh_4(n)pq_4(n)r_4(n)s_4(n) + 5040f_4(n)gh_4(n)r_4^3(n)$$

$$+ \ 1040 f_4(n) pq_4^3(n) s_4(n) + 16320 f_4(n) pq_4^2(n) r_4^2(n)$$

$$+ \ 1120 gh_4^3(n) r_4(n) s_4(n) + 3600 gh_4^2(n) pq_4^2(n) s_4(n)$$

$$+ \ 37560 gh_4^2(n) pq_4(n) r_4^2(n) + 47200 gh_4(n) pq_4^3(n) r_4(n)$$

$$+ \ 4344 pq_4^5(n) \ . \tag{5.14}$$

With the initial values given above, these equations can be solved by induction as in Lemma III.3.

Theorem V.2 The number of spanning trees on the four-dimensional Sierpinski gasket $SG_4(n)$ at stage n is given by

$$f_4(n) = 2^{\alpha_4(n)} 5^{\beta_4(n)} 7^{\gamma_4(n)} \ , \tag{5.15}$$

where the exponents are

$$\alpha_4(n) = \frac{3}{2}(5^n - 1) \ , \tag{5.16}$$

$$\beta_4(n) = \frac{3}{8}(5^{n+1} + 4n + 3) \ , \tag{5.17}$$

$$\gamma_4(n) = \frac{3}{8}(5^n - 4n - 1) \ . \tag{5.18}$$

$gh_4(n)$, $pq_4(n)$, $r_4(n)$ and $s_4(n)$ can also be expressed by these exponents:

$$gh_4(n) \ = \ 2^{\alpha_4(n)} 5^{\beta_4(n)-n-1} 7^{\gamma_4(n)+n} \ ,$$

$$pq_4(n) \ = \ 2^{\alpha_4(n)} 5^{\beta_4(n)-2n-2} 7^{\gamma_4(n)+2n} \ ,$$

$$r_4(n) \ = \ 2^{\alpha_4(n)} 5^{\beta_4(n)-3n-3} 7^{\gamma_4(n)+3n} \ ,$$

$$s_4(n) \ = \ 2^{\alpha_4(n)} 5^{\beta_4(n)-4n-3} 7^{\gamma_4(n)+4n} \ . \tag{5.19}$$

By the definition in eq. (2.1), we have the following corollary.

Corollary V.2 The asymptotic growth constant for SG_4 is given by

$$z_{SG_4} \ = \ \frac{3}{5} \ln 2 + \frac{3}{4} \ln 5 + \frac{3}{20} \ln 7$$

$$\simeq \ 1.914853265... \tag{5.20}$$

VI.   THE NUMBER OF SPANNING TREES ON $SG_d(n)$ FOR GENERAL d

With the above theorems that the numbers of spanning trees on $SG_2(n)$, $SG_3(n)$ and $SG_4(n)$ have only simple factors, we observe the following conjecture for the number of spanning trees $f_d(n) \equiv N_{ST}(SG_d(n))$ on $SG_d(n)$ for general d.

Conjecture VI.1 The number of spanning trees on the d-dimensional Sierpinski gasket $SG_d(n)$ at stage n is given by

$$f_d(n) = 2^{\alpha_d(n)}(d+1)^{\beta_d(n)}(d+3)^{\gamma_d(n)} , \qquad (6.1)$$

where the exponents are

$$\alpha_d(n) = \frac{d-1}{2}[(d+1)^n - 1] , \qquad (6.2)$$

$$\beta_d(n) = \frac{d-1}{2d}[(d+1)^{n+1} + dn + d - 1] , \qquad (6.3)$$

$$\gamma_d(n) = \frac{d-1}{2d}[(d+1)^n - dn - 1] . \qquad (6.4)$$

Notice that these exponents are positive integers when d is a positive integer and n is a non-negative integer. It is interesting that the recursion relations (4.3)-(4.5), (4.15)-(4.17), (5.1)-(5.4) and (5.10)-(5.14) are very complicated but the solutions are very simple. We do not know how to explain it. There may be a better method to solve this problem.

$SG_d(0)$ at state $n = 0$ is a complete graph with $(d+1)$ vertices, each of which is adjacent to all of the other vertices. Setting $n = 0$ into Conjecture VI.1 is consistent with the expectation that the number of spanning trees on the d-dimensional Sierpinski gasket $SG_d(0)$ at stage zero is given by $f_d(0) = (d+1)^{d-1}$.

With the number of vertices for $SG_d(n)$ given in eq. (2.6), Conjecture VI.1 leads to following result.

Conjecture VI.2 The asymptotic growth constant for $SG_d(n)$ is given by

$$z_{SG_d} = \frac{d-1}{d(d+1)}[d\ln 2 + (d+1)\ln(d+1) + \ln(d+3)] . \qquad (6.5)$$

TABLE I: Numerical values of z_{SG_d} and r_{SG_d}. The last digits given are rounded off.

d	D	k	z_{SG_d}	r_{SG_d}
2	1.585	4	1.048594857	0.862051042
3	2	6	1.569396409	0.928042816
4	2.322	8	1.914853265	0.953722370
5	2.585	10	2.172764568	0.966999152
6	2.807	12	2.378271274	0.974945363
7	3	14	2.548944395	0.980152960
8	3.170	16	2.694814686	0.983785240
9	3.322	18	2.822140640	0.986437211
10	3.459	20	2.935085659	0.988442577

Conjecture VI.1 and VI.2 reduce to Theorems III.1, V.1, V.2 and Corollaries III.2, V.1, V.2 respectively when d is set to two, three and four.

By eq. (2.4), z_{SG_d} has the upper bound $\ln b_{2d}$. It is of interest to see how close the value z_{SG_d} is to this bound. For this purpose, we define the ratio

$$r_{SG_d} = \frac{z_{SG_d}}{\ln b_{2d}} \ , \tag{6.6}$$

and list the first few values of z_{SG_d} and r_{SG_d} in Table I. Our results agree with the observation made in Ref. [11] that z_{SG_d} increases as the degree $k = 2d$ increases. Compared with the values $z_{\mathcal{L}_d}$ for d-dimensional hypercubic lattice $\mathcal{L}_d$ which also has $k = 2d$ in [11, 31], $z_{SG_d} < z_{\mathcal{L}_d}$ for all $d \geq 2$ indicates that SG_d is less densely connected than $\mathcal{L}_d$. The ratio r_{SG_d} increases with dimension and approaches one, but the convergence is not as fast as those for the regular lattices given in [11, 13].

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